Assignment 1

Goldstein 1.4 The equations of motion for the rolling disk are special cases of general linear differential equations of constraint of the form

$$\sum_{i=1}^{n} g_i(x_1, \ldots, x_n) \, dx_i = 0.$$ 

A constraint condition of this type is holonomic only if an integrating function, $f(x_1, \ldots, x_2)$ can be found that turns it into an exact differential. Clearly the function must be such that

$$\frac{\partial (fg_i)}{\partial x_j} = \frac{\partial (fg_j)}{\partial x_i}$$

for all $i \neq j$. Show that no such integrating factor can be found for either of Eqs. (1.39).

The differential constraint above can be written out as

$$0 = g_x \, dx + g_y \, dy + \cdots$$

This would be integrable if there were some function, $f(x, y, \cdots)$, which, which multiplying this turned it into a perfect differential, i.e

$$0 = dF = \frac{\partial F}{\partial x} \, dx + \frac{\partial F}{\partial y} \, dy + \cdots$$

$$= f \cdot g_x \, dx + f \cdot g_y \, dy + \cdots$$

In such a case, we would actually have a holonomic constraint, $F = $ constant, i.e. an algebraic relation between the coordinates. To be able to have an integrating factor, $f$, it must be the case that the mixed partial derivatives must be equal, i.e

$$\frac{\partial (fg_x)}{\partial y} = \frac{\partial (fg_y)}{\partial x}$$

and so on for all the mixed partial derivatives. In Goldstein’s notation, this is

$$\frac{\partial (f g_i)}{\partial x_j} = \frac{\partial (f g_j)}{\partial x_i}$$

for $i \neq j$.

Applying this to

$$0 = dx - a \sin \theta \, d\phi$$

we note $g_x = 1, g_y = 0, g_\theta = 0$ and $g_\phi = -a \sin \theta$. We have a bunch of conditions:

$$\frac{\partial (fg_x)}{\partial y} = \frac{\partial (fg_y)}{\partial x}$$

$$\frac{\partial (fg_\theta)}{\partial \phi} = \frac{\partial (fg_\phi)}{\partial \theta}$$

$$\frac{\partial (fg_\phi)}{\partial \theta} = \frac{\partial (fg_\phi)}{\partial \phi}$$
These reduce to

\[ \frac{\partial f}{\partial y} = 0 \]
\[ \frac{\partial f}{\partial \theta} = 0 \]
\[ \frac{\partial f}{\partial \phi} = -a \sin \theta \frac{\partial f}{\partial x} \]
\[ \frac{\partial (f \sin \theta)}{\partial \theta} = 0 \]

Note that the final equation can be written

\[ f \cos \theta + \sin \theta \frac{\partial f}{\partial \theta} = 0 \]

Thus, unless \(\theta = \pi/2\), the only solution to the above equations is \(f = 0\) and no integrating factor exists. For the case that \(\theta = \pi/2\), the disk is rolling parallel to the \(x\)-axis and \(dx = ad\phi\) which can be integrated to \(x = a(\phi - \phi_0)\) where \(\phi_0\) is a constant.

Applying this to

\[ 0 = dy + a \cos \theta d\phi \]

we get \(g_x = 0, g_y = 1, g_\theta = 0,\) and \(g_\phi = a \cos \theta\). This time the conditions become

\[ \frac{\partial f}{\partial x} = 0 \]
\[ \frac{\partial f}{\partial \theta} = 0 \]
\[ \frac{\partial f}{\partial \phi} = a \cos \theta \frac{\partial f}{\partial y} \]
\[ -\sin \theta f + \cos \theta \frac{\partial f}{\partial \theta} = 0 \]

Again, unless \(\theta = 0\), the only solution to these equations is \(f = 0\). So no integrating factor exists in general to render this a holonomic constraint.
Goldstein 1.5 Two wheels of radius $a$ are mounted on the ends of a common axle of length $b$ such that the wheels rotate independently. The whole combination rolls without slipping on a plane. Show that there are two nonholonomic equations of constraint,
\[
\cos \theta \, dx + \sin \theta \, dy = 0 \\
\sin \theta \, dx - \cos \theta \, dy = \frac{1}{2} \, a \left( d\phi + d\phi' \right)
\]
(where $\theta$, $\phi$, and $\phi'$ have meanings similar to those in the problem of a single vertical disk, and $(x, y)$ are the coordinates of a point on the axle midway between the two wheels) and one holonomic equation of constraint,
\[
\theta = C - \frac{a}{b} \left( \phi - \phi' \right),
\]
where $C$ is a constant.

As in the previous problem with one wheel, each wheel here of the pair satisfies the same constraints as a single rolling disk. We have two wheels so we have two sets of equations in obvious notation:
\[
\begin{align*}
    dx_1 - a \sin \theta \, d\phi_1 &= 0 \\
    dy_1 + a \cos \theta \, d\phi_1 &= 0 \\
    dx_2 - a \sin \theta \, d\phi_2 &= 0 \\
    dy_2 + a \cos \theta \, d\phi_2 &= 0
\end{align*}
\]
The center of the axle can be considered the center of mass so that
\[
\begin{align*}
    x &= \frac{1}{2} (x_1 + x_2) \\
    y &= \frac{1}{2} (y_1 + y_2)
\end{align*}
\]
so that we can reduce our four equations to two:
\[
\begin{align*}
    dx &= \frac{a}{2} \sin \theta \left( d\phi_1 + d\phi_2 \right) \\
    dy &= \frac{a}{2} \cos \theta \left( d\phi_1 + d\phi_2 \right)
\end{align*}
\]
Through two combinations involving $\sin \theta$ and $\cos \theta$, we can write these as
\[
\begin{align*}
    \cos \theta \, dx + \sin \theta \, dy &= 0 \\
    \sin \theta \, dx - \cos \theta \, dy &= \frac{a}{2} \left( d\phi_1 + d\phi_2 \right)
\end{align*}
\]
It may look like we are done, but what we have constrained is that the wheels roll in unison. But our approach so far, leaves open the possibility that they could, in fact, change their distance between them. To fix the constraint that the distance between the wheels remains at a constant value, $b$, such that $x_2 - x_1 = b \cos \theta$ and $y_2 - y_1 = b \sin \theta$, we impose
\[
\dot{x}_2 - \dot{x}_1 = -b \dot{\theta} \sin \theta
\]
Using the relations between the $x$’s and the $\phi$’s, we can write
\[
\dot{x}_2 - \dot{x}_1 = a \sin \theta (\dot{\phi}_2 - \dot{\phi}_1)
\]
Equating these, dropping the $\sin \theta$, and integrating we have
\[
\theta = C - \frac{a}{b} \left( \phi_2 - \phi_1 \right)
\]
Goldstein 1.8 If \( L \) is a Lagrangian for a system of \( n \) degrees of freedom satisfying Lagrange’s equations, show by direct substitution that

\[
L' = L + \frac{dF(q_1, \ldots, q_n, t)}{dt}
\]

also satisfies Lagrange’s equations where \( F \) is any arbitrary, but differentiable, function of its arguments.

We know \( L \) satisfies Lagrange’s equations and we need only show the same for \( L' \). To that end, we note that we can write

\[
\frac{dF}{dt} = \frac{\partial F}{\partial t} + \sum_{i=1}^{n} \frac{\partial F}{\partial q_i} \dot{q}_i
\]

so that we can also say

\[
\frac{\partial}{\partial q_j} \left( \frac{dF}{dt} \right) = \frac{\partial}{\partial t} \left( \frac{\partial F}{\partial q_j} \right) + \sum_{i=1}^{n} \frac{\partial}{\partial q_i} \left( \frac{\partial F}{\partial q_j} \right) \dot{q}_i
\]

\[
= \frac{d}{dt} \left( \frac{\partial F}{\partial q_j} \right)
\]

Further, because \( \dot{F} \) is independent of the \( \dot{q}_i \), we can conclude

\[
\frac{\partial \dot{F}}{\partial \dot{q}_i} = \frac{\partial F}{\partial q_i}
\]

Constructing Lagrange’s equations in terms of \( L' \), we get

\[
\frac{d}{dt} \left( \frac{\partial L'}{\partial \dot{q}_i} \right) - \frac{\partial L'}{\partial q_i} = \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{q}_i} \right) - \frac{\partial L}{\partial q_i} + \frac{d}{dt} \left( \frac{dF}{dt} \right) - \frac{\partial}{\partial q_i} \left( \frac{dF}{dt} \right)
\]

The first two terms on the right vanish because \( L \) satisfies Lagrange’s equations and the third and fourth terms can be written

\[
\frac{d}{dt} \left( \frac{\partial L'}{\partial \dot{q}_i} \right) - \frac{\partial L'}{\partial q_i} = \frac{d}{dt} \left[ \frac{\partial F}{\partial q_i} \right] - \frac{\partial}{\partial q_i} \left( \frac{dF}{dt} \right)
\]

which, of course, vanishes. Thus \( L' \) as a shift of \( L \) by a total derivative also satisfies Lagrange’s equations.
Goldstein 1.10 Let $q_1, \ldots, q_n$ be a set of independent generalized coordinates for a system of $n$ degrees of freedom, with a Lagrangian $L(q, \dot{q}, t)$. Suppose we transform to another set of independent coordinates $s_1, \ldots, s_n$ by means of transformation equations

$$q_i = q_i(s_1, \ldots, s_n, t), \quad i = 1, \ldots, n.$$  

(Such a transformation is called a point transformation.) Show that if the Lagrangian function is expressed as a function of $s_j, \dot{s}_j, t$ through the equations of transformation, then $L$ satisfies Lagrange’s equations with respect to the $s$ coordinates:

$$\frac{d}{dt} \left( \frac{\partial L}{\partial \dot{s}_j} \right) - \frac{\partial L}{\partial s_j} = 0.$$

In other words, the form of Lagrange’s equations is invariant under a point transformation.

Again, we have to be careful when we talk about what various quantities depend on. In particular we are assuming that there are $n$ coordinates, $q_i$, that depend on new coordinates $s_i$ and $t$. We also assume that the inverse relation exists, namely

$$s_i = s_i(q_j, t)$$

Note, importantly, that a point transformation does not depend on the derivatives of the new coordinates, namely $\dot{s}_i$. However, we do assume that the Lagrangian can depend on the derivatives of the new coordinates.

Explicitly, we can write

$$L = L(q_i(s_j, t), \dot{q}_i(s_j, \dot{s}_j, t), t)$$

If we now write the Lagrange equations, we have

$$0 = \frac{\partial L}{\partial \dot{q}_i} \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{q}_i} \right) - \frac{\partial L}{\partial q_i}$$

where we have used the chain rule twice because the derivatives of $L$ depend on both $s_j$ and $\dot{s}_j$.

From a couple of derivations identical to those in problem 1.8, we can establish that

$$\frac{\partial \dot{s}_j}{\partial \dot{q}_i} = \frac{\partial s_j}{\partial \dot{q}_i}$$

and

$$\frac{d}{dt} \left( \frac{\partial s_j}{\partial q_i} \right) = \frac{\partial \dot{s}_j}{\partial q_i}.$$

Using these, we can simplify our Lagrange equations as

$$0 = \sum_{j=1}^{n} \frac{\partial L}{\partial \dot{s}_j} \frac{\partial s_j}{\partial q_i} + \frac{\partial L}{\partial \dot{s}_j} \frac{d}{dt} \left( \frac{\partial s_j}{\partial q_i} \right) - \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{s}_j} \frac{\partial s_j}{\partial q_i} + \frac{\partial L}{\partial s_j} \frac{\partial \dot{s}_j}{\partial q_i} \right)$$

Note that the first term in square brackets vanishes because $s_j$ does not depend on the velocities so that

$$\frac{\partial s_j}{\partial \dot{q}_i} = 0$$

Expanding the final term in a product rule and rearranging a bit, we have

$$0 = \sum_{j=1}^{n} \frac{\partial L}{\partial \dot{s}_j} \frac{\partial s_j}{\partial q_i} + \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{s}_j} \frac{\partial s_j}{\partial q_i} \right) + \frac{\partial L}{\partial \dot{s}_j} \frac{d}{dt} \left( \frac{\partial s_j}{\partial q_i} \right) - \frac{\partial L}{\partial s_j} \frac{d}{dt} \left( \frac{\partial \dot{s}_j}{\partial q_i} \right)$$

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The final two terms cancel and we have
\[ 0 = \sum_{j=1}^{n} \left\{ \frac{\partial L}{\partial s_j} - \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{s}_j} \right) \right\} \frac{\partial s_j}{\partial q_i} \]

Provided the new coordinates are independent, their derivatives with respect to the old coordinates will be invertible and the only way this can vanish is if the quantity in curly braces vanishes for each \( j \). This of course is just Lagrange’s equations in the new coordinates. Thus Lagrange’s equations are invariant in form under a point transformation.

Goldstein 1.12 The escape velocity of a particle on Earth is the minimum velocity required at Earth’s surface in order that the particle can escape from Earth’s gravitational field. Neglecting the resistance of the atmosphere, the system is conservative. From the conservation theorem for potential plus kinetic energy show that the escape velocity for Earth, ignoring the presence of the Moon, is 11.2 km/s.

This is a straightforward recall of a problem from introductory physics. For a particle of mass \( m \), conservation is just
\[ E = V + T = \frac{1}{2} mv^2 - \frac{GM_e m}{r} \]
where everything has the usual meaning, i.e. \( E \) is the constant “total” energy, \( G \) is Newton’s constant, \( M_e \) is the mass, \( r \) is the radial distance of the particle from the center of the earth, and \( v \) is the velocity of the particle. The notion of the escape velocity, \( v_{esc} \), or a minimum velocity suggests that this velocity is exactly that which would allow the particle to have no kinetic energy an infinite distance from the source. In this case, the value of \( E \) would be zero and we have
\[ 0 = \frac{1}{2} mv_{esc}^2 - \frac{GM_e m}{R_e} \]
Solving this for \( v_{esc} \) and putting in numbers, we have
\[ v_{esc} = \sqrt{\frac{2GM_e}{R_e}} \approx 11.2 \text{ km/s} \]
Goldstein 1.13 Rockets are propelled by the momentum reaction of the exhaust gases expelled from the tail. Since these gases arise from the reaction of the fuels carried in the rocket, the mass of the rocket is not constant, but decreases as the fuel is expended. Show that the equation of motion for a rocket projected vertically upward in a uniform gravitational field, neglecting atmospheric friction, is

\[ m \frac{dv}{dt} = -v' \frac{dm}{dt} - mg, \]

where \( m \) is the mass of the rocket and \( v' \) is the velocity of the escaping gases relative to the rocket. Integrate this equation to obtain \( v \) as a function of \( m \), assuming a constant time rate of loss of mass. Show, for a rocket starting initially from rest, with \( v' \) equal to 2.1 km/s and a mass loss per second equal to 1/60th of the initial mass, that in order to reach the escape velocity the ratio of the weight of the fuel to the weight of the empty rocket must be almost 300!

Consider the rocket (and its unburned fuel) at some time \( t \) moving with respect to the earth with a vertical velocity of \( v \) and momentum \( mv \). At a later time \( t + dt \) there are two parts to the system, namely the rocket (and some still unburned fuel) of mass \( m - dm \) moving with \( v + dv \) and the expelled fuel of mass \( dm \) which moves with velocity \( v_f \) with respect to the earth. The total momentum of the system at the later time is

\[ p + dp = (v + dv)(m - dm) + v_f dm = p + m dv + (v_f - v) dm + \cdots \]

where we ignore the higher order terms. Thus we have

\[ -mg = \frac{dp}{dt} = m \frac{dv}{dt} + v' \frac{dm}{dt} \]

where \( v' = v_f - v \) is the speed of the expelled fuel with respect to the rocket.

If we assume \( v' \) is a constant, we can integrate this equation in \( t \) to get

\[ v(t) = -gt - v' \ln m(t) + C \]

where \( C = v_0 + v' \ln m_0 \) where \( v_0 \) and \( m_0 \) are the initial speed and mass of the rocket, respectively. However, we want the velocity as a function of \( m \). We can do this by writing

\[ \frac{dv}{dm} = -\frac{g}{\dot{m}} - \frac{v'}{m} \]

assuming \( v' \) and \( \dot{m} \) are constants and integrating to get

\[ v(m) = v(m_0) + \frac{g}{\dot{m}} (m_0 - m) - v' \ln \left( \frac{m}{m_0} \right) \]

Assuming the initial speed is zero, we can write this in terms of the values given, namely \( v' = 2.1 \) km/s, \( \dot{m} = -m_0/60s \) and the escape velocity:

\[ 11200 = -60 \cdot 9.8 \left(1 - x\right) - 2100 \ln(x) \]

where \( x = m/m_0 \). Using Mathematica or Maple to solve this equation, we find \( x \approx 0.003653 \). Thus the initial mass of the rocket on the launch pad must be about 275 times the mass of the rocket (presumably empty) at escape velocity.
Goldstein 1.20 A particle of mass \( m \) moves in one dimension such that it has the Lagrangian

\[
L = \frac{1}{12} m^2 \dot{x}^4 + m \dot{x}^2 V(x) - V^2(x)
\]

where \( V \) is some differentiable function of \( x \). Find the equation of motion for \( x(t) \) and describe the physical nature of the system on the basis of the equation.

The Lagrange equations become

\[
0 = \frac{\partial L}{\partial x} - \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{x}} \right)
= m \dot{x}^2 V'(x) - 2V(x)V'(x) - \frac{d}{dt} \left( \frac{1}{3} m^2 \dot{x}^3 + 2m \dot{x} V(x) \right)
= 2V'(x) \left( \frac{1}{2} m \dot{x}^2 - V(x) \right) - 2m \left( \frac{1}{2} m \dot{x}^2 \ddot{x} + \ddot{x} V(x) + \dot{x}^2 V'(x) \right)
= -2 \left( \frac{1}{2} m \dot{x}^2 + V(x) \right) \left( m \dddot{x} + V'(x) \right)
\]

The right hand side must vanish. We thus have two possibilities. Interpreting \( x \) as the generalized one dimensional coordinate and \( V(x) \) as a one dimensional potential, either the sum of the kinetic and potential energies must vanish or the corresponding force is conservative and a particle of mass \( m \) moves along the \( x \) direction under the influence of the conservative force given by \(-V'(x)\).
Goldstein 1.21 Two mass points of mass \( m_1 \) and \( m_2 \) are connected by a string passing through a hole in a smooth table so that \( m_1 \) rests on the table surface and \( m_2 \) hangs suspended. Assuming \( m_2 \) moves only in a vertical line, what are the generalized coordinates for the system? Write the Lagrange equations for the system, and if possible, discuss the physical significance any of them might have. Reduce the problem to a single second-order differential equation and obtain a first integral of the equation. What is its physical significance? (Consider the motion only until \( m_1 \) reaches the hole.)

The system would have three degrees of freedom if the problem were that \( m_1 \) moves freely on the table and \( m_2 \) only moves up and down. However, we will assume that the string between the two mass remains taut and that the radial position of \( m_1 \) is determined by the vertical height of \( m_2 \) (and vice versa). Thus there are two degrees of freedom. Generalized coordinates can be thought of as the two coordinates in the plane for \( m_1 \), \( r \) and \( \theta \). The vertical distance from the hole for \( m_2 \) is given by \( x_0 - r \) where \( x_0 \) is the length of the string. If we set the zero of potential to be when \( m_1 \) is on the hole, we have the Lagrangian

\[
L = \frac{1}{2} m_1 \left( \dot{r}^2 + r^2 \dot{\theta}^2 \right) + \frac{1}{2} m_2 (\dot{x}_0 - \dot{r})^2 - m_2 g [x_0 - (x_0 - r)]
\]

The equations of motion from this Lagrangian become

\[
0 = (m_1 r \ddot{\theta}^2 - m_2 g) - (m_1 + m_2) \ddot{r}
\]

\[
0 = 0 - m_1 \frac{d}{dt}(r^2 \dot{\theta})
\]

The second equation can be immediately integrated and yields conservation of angular momentum:

\[
\ell = m_1 r^2 \dot{\theta}
\]

Having integrated one equation, we can use this result to eliminate \( \dot{\theta} \) in the first equation. We have,

\[
(m_1 + m_2) \ddot{r} = \frac{\ell^2}{m_1 r^3} - m_2 g
\]

There is a further integral of the motion that we can obtain by multiplying this equation by \( \dot{r} \) and integrating in time:

\[
(m_1 + m_2) \ddot{r} \dot{r} = \frac{\ell^2 \dot{r}}{m_1 r^3} - m_2 g \dot{r}
\]

\[
\frac{1}{2} (m_1 + m_2) r^2 = -\frac{\ell^2}{2m_1} \frac{1}{r^2} - m_2 gr + C_0
\]

On rearranging this a bit, one can see that this is a statement of conservation of energy.
Goldstein 1.22 Obtain the Lagrangian and equations of motion for the double pendulum illustrated in Fig. 1.4, where the lengths of the pendula are \( l_1 \) and \( l_2 \) with corresponding masses \( m_1 \) and \( m_2 \).

We can work with two pairs of coordinates for the two masses, \((x_1, y_1)\) and \((x_2, y_2)\). They are not independent, of course. We can write these Cartesian coordinates (with an assumed origin at the top pivot point) in terms of the (generalized) angular coordinates given in the figure:

\[
\begin{align*}
x_1 &= l_1 \sin \theta_1 \\
y_1 &= -l_1 \cos \theta_1 \\
x_2 &= x_1 - l_2 \sin \theta_2 \\
y_2 &= y_1 - l_2 \cos \theta_2
\end{align*}
\]

The Lagrangian is given by

\[
L = \frac{1}{2} m_1 (\dot{x}_1^2 + \dot{y}_1^2) + \frac{1}{2} m_2 (\dot{x}_2^2 + \dot{y}_2^2) - m_1 g y_1 - m_2 g y_2
\]

\[
= \frac{1}{2} (m_1 + m_2) l_1^2 \dot{\theta}_1^2 + \frac{1}{2} m_2 l_2^2 \dot{\theta}_2^2 - m_2 l_1 l_2 \dot{\theta}_1 \dot{\theta}_2 \cos(\theta_1 + \theta_2) + (m_1 + m_2) gl_1 \cos \theta_1 + m_2 gl_2 \cos \theta_2
\]

From this Lagrangian, we get the equations of motion

\[
\begin{align*}
(m_1 + m_2) l_1^2 \ddot{\theta}_1 - m_2 l_1 l_2 \dot{\theta}_2 \cos(\theta_1 + \theta_2) + m_2 l_1 l_2 \dot{\theta}_1 \dot{\theta}_2 \sin(\theta_1 + \theta_2) + (m_1 + m_2) gl_1 \sin \theta_1 &= 0 \\
m_2 l_2^2 \ddot{\theta}_2 - m_2 l_1 l_2 \dot{\theta}_1 \cos(\theta_1 + \theta_2) + m_2 l_1 l_2 \dot{\theta}_1^2 \sin(\theta_1 + \theta_2) + m_2 gl_2 \sin \theta_2 &= 0
\end{align*}
\]

Note that some constants in both can be factored out.

Goldstein 1.23 Obtain the equation of motion for a particle falling vertically under the influence of gravity when frictional forces obtainable from a dissipation function \( kv^2 / 2 \) are present. Integrate the equation to obtain the velocity as a function of time and show that the maximum possible velocity for a fall from rest is \( v = mg/k \).

The Lagrangian is

\[
L = T - V = \frac{1}{2} m \dot{z}^2 - mg z
\]

Lagrange’s equation is

\[
0 = \frac{d}{dt} \frac{\partial L}{\partial \dot{z}} - \frac{\partial L}{\partial z} + \frac{\partial F}{\partial \dot{z}} = m \ddot{z} + mg + kv
\]

As an equation for \( \dot{v} = \dot{z} \), this is

\[
m \dot{v} + kv + mg = 0
\]

This can be solved with an integrating factor, \( e^{kt/m} \), so that

\[
\frac{d}{dt} \left( me^{kt/m} \dot{v} \right) = -mg e^{kt/m}
\]

Integrating and rearranging, we get

\[
v(t) = e^{-kt/m} \left[ -\frac{mg}{k} e^{kt/m} + C \right]
\]

The integration constant can be found from the initial condition \( v(0) = v_0 \), so the solution becomes

\[
v(t) = v_0 e^{-kt/m} + \frac{mg}{k} \left[ e^{-kt/m} - 1 \right]
\]

The maximum possible velocity is obtained as \( t \to \infty \). This leads to \( v \to -mg/k \). The minus sign denotes only that we are falling in the negative \( z \) direction.